scheme is advantageous for problems in which minimum mixing of the marks at each step is important.

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Chebyshev Approximations to the Gamma Function

By Helmut Werner and Robert Collinge

In this note several Chebyshev approximations are given for the function $y = \Gamma(x + 2)$ for x in the $0 \leqq x \leqq 1.0$ range. The approximations were obtained from a table of $\Gamma(x + 2)$, employing well-known methods as described in numerous papers; see for instance [1] and the literature quoted there. The table of $\Gamma(x + 2)$ was calculated from the asymptotic expansion of log $\Gamma(z)$ as given in [2] to provide data accurate to at least 10^{-21}. Compare also [3].

The asymptotic expansion of ln $\Gamma(z)$, is given by

$$\ln \Gamma(z) = (z - \tfrac{1}{2}) \ln z - z + \ln \sqrt{2\pi} + \Phi(z)$$

where

$$\Phi(z) = \sum_{r=1}^{n} \frac{(-1)^{r-1} B_r}{2r(2r - 1)} \frac{1}{z^{2r-1}} + R_n(z),$$

and B_r is the rth Bernoulli number.

It can be shown [2] that for $z > 0$ the value of $\Phi(z)$ always lies between the sum of n terms and the sum of $(n + 1)$ terms of the series, for all values of n. In terminating this series with the nth term the error $R_n(z)$ will be less than

$$\frac{B_{n+1}}{2(n + 1)(2n + 1)} \cdot \frac{1}{z^{2n+1}}.$$

By truncating $\Phi(z)$ at the 10th term it is easily shown that for values of $z \geqq 13$, the error in the expansion is less than 5.5×10^{-22}. We therefore replace $\Phi(z)$ by $\sum_{i=1}^{10} A_i/z^{2i-1}$ and calculate ln $\Gamma(z)$ for values of z in the range $13 \leqq z \leqq 14$.

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TABLE 1

Table of Coefficients

n	7	8	10
$\epsilon_{max}^{(n)}$	0.25×10^{-7}	0.16×10^{-9}	0.74×10^{-11}
$\nu = 0$	0.99999 99758	0.99999 99998 452	0.99999 99999 9269
1	0.42278 74605	0.42278 43662 730	0.42278 43369 6202
2	0.41177 41955	0.41183 92935 920	0.41184 02517 9616
3	0.08211 17404	0.08159 03449 474	0.08157 82187 8492
4	0.07211 01567	0.07416 00915 535	0.07423 79076 0629
5	0.00445 11400	0.00007 55964 181	−0.00021 09074 6731
6	0.00515 89951	0.01033 20685 065	0.01097 36958 4174
7	0.00160 63118	−0.00157 80074 635	−0.00246 67479 8054
8		0.00079 62464 760	0.00153 97681 0472
9			−0.00034 42342 0456
10			0.00006 77105 7117

n	13	15
$\epsilon_{max}^{(n)}$	0.96×10^{-14}	0.97×10^{-16}
$\nu = 0$	0.99999 99999 99990 44	0.99999 99999 99999 9032
1	0.42278 43351 02334 79	0.42278 43350 98518 1178
2	0.41184 03301 66781 29	0.41184 03304 21981 4831
3	0.08157 69261 24155 46	0.08157 69194 01388 6786
4	0.07424 89154 19444 74	0.07424 90079 43401 2692
5	−0.00026 61865 94953 06	−0.00026 69510 28755 5266
6	0.01114 97143 35778 93	0.01115 38196 71906 6992
7	−0.00283 64625 20372 82	−0.00285 15012 43034 6494
8	0.00206 10918 50225 54	0.00209 97590 35077 0629
9	−0.00083 75646 85135 17	−0.00090 83465 57420 0521
10	0.00037 53650 52263 07	0.00046 77678 11496 4956
11	−0.00012 14173 48706 32	−0.00020 64476 31915 9326
12	0.00002 79832 88993 83	0.00008 15530 49806 6373
13	−0.00000 30301 90810 28	−0.00002 48410 05384 8712
14		0.00000 51063 59207 2582
15		−0.00000 05113 26272 6698

n	17	18
$\epsilon_{max}^{(n)}$	0.10×10^{-17}	0.10×10^{-18}
$\nu = 0$	0.99999 99999 99999 99901 2	0.99999 99999 99999 99990 02
1	0.42278 43350 98467 79580 6	0.42278 43350 98467 21319 64
2	0.41184 03304 26367 20638 1	0.41184 03304 26430 62304 23
3	0.08157 69192 50260 90508 9	0.08157 69192 47528 84581 87
4	0.07424 90106 80090 41696 9	0.07424 90107 42094 91715 38
5	−0.00026 69810 33348 38176 8	−0.00026 69818 88740 38315 07
6	0.01115 40360 24034 39169 2	0.01115 40438 29069 91793 28
7	−0.00285 25821 44619 65607 6	−0.00285 26318 64702 11862 89
8	0.00210 36287 02459 83329 2	0.00210 38579 20672 20524 09
9	−0.00091 84843 69099 08014 2	−0.00091 92675 95039 95026 11
10	0.00048 74227 94476 75810 4	0.00048 94361 06998 14458 34
11	−0.00023 47204 01891 94985 9	−0.00023 86428 33752 63647 10
12	0.00011 15339 51966 59947 0	0.00011 73283 10224 09396 51
13	−0.00004 78747 98383 44672 4	−0.00005 43183 86280 13508 99
14	0.00001 75102 72717 90508 0	0.00002 28140 41153 66022 75
15	−0.00000 49203 75090 42313 2	−0.00000 80523 43363 48309 46
16	0.00000 09199 15640 71621 4	0.00000 21741 77495 45532 64
17	−0.00000 00839 94049 59039 7	−0.00000 03889 70057 38769 55
18		0.00000 00339 81801 01810 43

For the convenience of the reader the A_i coefficients are quoted below, to 25 significant figures.

$$
\begin{aligned}
A_1 &= 0.08333\ 33333\ 33333\ 33333\ 33333\ 3 \\
A_2 &= -0.00277\ 77777\ 77777\ 77777\ 77777\ 78 \\
A_3 &= 0.00079\ 36507\ 93650\ 79365\ 07936\ 508 \\
A_4 &= -0.00059\ 52380\ 95238\ 09523\ 80952\ 381 \\
A_5 &= 0.00084\ 17508\ 41750\ 84175\ 08417\ 508 \\
A_6 &= -0.00191\ 75269\ 17526\ 91752\ 69175\ 27 \\
A_7 &= 0.00641\ 02564\ 10256\ 41025\ 64102\ 56 \\
A_8 &= -0.02955\ 06535\ 94771\ 24183\ 00653\ 6 \\
A_9 &= 0.17964\ 43723\ 68830\ 57316\ 49385 \\
A_{10} &= -1.39243\ 22169\ 05901\ 11642\ 7432 \\
\ln\sqrt{2\pi} &= 0.91893\ 85332\ 04672\ 74178\ 03297
\end{aligned}
$$

A triple precision logarithm routine was used to evaluate $\ln z$, and then an exponential routine to calculate $\Gamma(z) = e^{\ln \Gamma(z)}$. Each of these routines produces results accurate to at least 24 significant digits.

After obtaining a table of $\Gamma(z)$ for z in the range $13 \leqq z \leqq 14$, we made use of the recursion formula $\Gamma(z + 1) = z\Gamma(z)$ in order to obtain a table of $\Gamma(x + 2)$ for x in the range $0 \leqq x \leqq 1.0$.

From the tests made on the results obtained, the values of $\Gamma(x + 2)$ were shown to be accurate to at least 21 significant figures.

Several Chebyshev approximations have been calculated to provide varying degrees of accuracy. Let

$$\Gamma(2 + x) = \sum_{\nu=0}^{n} a_\nu^{(n)} x^\nu + \epsilon_n(x)$$

and

$$\epsilon_{\max}^{(n)} = \max_{0 \leqq x \leqq 1} | \epsilon_n(x) |.$$

Table 1 gives the coefficients $a_\nu^{(n)}$ for $n = 7, 8, 10, 13, 15, 17, 18$ together with the corresponding $\epsilon_{\max}^{(n)}$.

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